Multiplication Table

Levi-Civita

∅	1	i
1	1	i
i	i	−1

$$(u_1 + iu_2)^2 = u_1^2 - u_2^2 + i(2u_1u_2)$$

	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

$$-1 = i^2 = j^2 = k^2 = ijk$$

$$(u_1 + iu_2 + ju_3 + ku_4)^2$$

Transformation Matrix $L(u)$, with $L(u)\cdot u = u^2$

Quaternion

$$\begin{pmatrix} u_1 & -u_2 \\ +u_2 & u_1 \end{pmatrix}$$

$$L(\vec{u})\cdot\vec{u} = \vec{r}$$

$$2L(\vec{u})\dot{\vec{u}} = \dot{\vec{r}}$$

$$\begin{pmatrix} u_1 & -u_2 & -u_3 & -u_4 \\ u_2 & u_1 & u_4 & -u_3 \\ u_3 & -u_4 & u_1 & u_2 \\ u_4 & u_3 & -u_2 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_1^2 - u_2^2 - u_3^2 - u_4^2 \\ 2u_1u_2 \\ 2u_1u_3 \\ 2u_1u_4 \end{pmatrix}$$

Multiplication Table

	1	i	j	k
1	1	i	j	-k
i	i	-1	k	j
j	j	-k	-1	-i
k	k	j	-i	1

$k^2 = +1!$

Kustaanheimo+Stiefel

$$\begin{pmatrix} u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ 2u_1u_2 - 2u_3u_4 \\ 2u_1u_3 + 2u_2u_4 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

1. TWO-BODY PROBLEM

Hamiltonian $H = T + V$ of 2-b-Pr.

$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} - \frac{G m_1 m_2}{r} = H_{cm} + H_{rel}$$

$\vec{p}_1 = m_1 \vec{v}_1$; $\vec{p}_2 = m_2 \vec{v}_2$; $r = |\vec{r}|$

Move to <u>relative</u> coordinates

$\vec{v} = \vec{v}_1 - \vec{v}_2$; $\vec{r} = \vec{r}_1 - \vec{r}_2$; $\vec{p} = \mu \vec{v}$

$\mu = \frac{m_1 m_2}{m_1 + m_2}$; $M = m_1 + m_2$; $\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$

$\vec{v}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$; $\vec{p}_{cm} = M \vec{v}_{cm} = \vec{p}_1 + \vec{p}_2$

$$H_{cm} = \frac{\vec{p}_{cm}^{\,2}}{2M} \qquad H_{rel} = \frac{\vec{p}^{\,2}}{2\mu} - \frac{G m_1 m_2}{r}$$

Canonical Equations:

$$\dot{\vec{p}}_{cm} = -\frac{\partial H_{cm}}{\partial \vec{r}_{cm}} = 0 \Rightarrow \vec{v}_{cm} = \text{const. (3 const.)}$$

$$\dot{\vec{r}}_{cm} = \frac{\partial H_{cm}}{\partial \vec{p}_{cm}} = \frac{\vec{p}_{cm}}{M} = \vec{v}_{cm} \text{ (Def of } \vec{p}_{cm})$$

Conservation law: $\vec{r}_{cm} \times \vec{v}_{cm} = \text{const.}$ $\vec{v}_{cm\,1,2,3}$ $r_1 v_2 - r_2 v_1$ $r_3 v_1 - r_1 v_3$ $r_2 v_3 - r_3 v_2$

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Relative Motion

(i) $\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = \frac{G m_1 m_2}{r^2}\hat{r} = \mu \dot{\vec{v}}$

$\dot{\vec{v}} = \frac{GM}{r^2}\hat{r}$

(ii) $\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p}}{\mu} = \vec{v}$

$$H_{rel} = \frac{\vec{p}^2}{2\mu} - \frac{G m_1 m_2}{r}$$

Some conserved quantities:

- $E = \frac{1}{2}\mu v^2 - \frac{G m_1 m_2}{r}$; $\frac{E}{\mu} = \frac{1}{2}v^2 - \frac{GM}{r}$

- Specific angular momentum: $\vec{j} = \vec{r} \times \vec{v}$

$$\frac{d}{dt}\vec{j} = \frac{d}{dt}\vec{r}\times\vec{v} = \vec{v}\times\vec{v} + \vec{r}\times\dot{\vec{v}} = -\frac{GM}{r^3}\vec{r}\times\vec{r} = 0$$

- Runge-Lenz (eccentricity vector)

$$\vec{e} = \frac{\vec{v}\times\vec{j}}{GM} - \frac{\vec{r}}{r}$$ is constant of motion and $|\vec{e}| = e$ ecc.

(Note: not all of $E, \vec{j}, \vec{e}$ are isolating!
not all are even independent!)

Proof: $\frac{d\vec{e}}{dt} \overset{?}{=}$ | $-\frac{\vec{r}\times\vec{j}}{r^3} - \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\vec{j}\times\vec{r}}{r^3} - \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$

$$\frac{\vec{j}\times\vec{r}}{r^3} = \frac{(\vec{r}\times\vec{v})\times\vec{r}}{r^3} = \frac{\vec{v}\,r^2 - \vec{r}(\vec{v}\vec{r})}{r^3}$$

$$(A\times B)\times C = B(AC) - A(BC)$$

$$= \frac{\vec{v}}{r} - (\vec{v}\vec{r})\frac{\vec{r}}{r^3} \overset{!}{=} \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

3a

$$\frac{d}{dt}\frac{\vec{r}}{r} \quad : \quad \frac{x}{r}, \qquad r = \sqrt{x^2+y^2+z^2}$$

$$\frac{d}{dt}\frac{x}{r} = \frac{v_x}{r} - \frac{x}{r^2}\cdot\frac{dr}{dt}$$

$$= \frac{v_x}{r} - \frac{x}{r^2}\,\frac{\cancel{2}\cdot(x\cdot v_x + y\cdot v_y + z\cdot v_z)}{\cancel{2}\cdot r^3} \quad \checkmark$$

Isolating Integrals:

- generally only E
- sph. symm. pot. $E, \vec{J}$
- axi symm. pot. $E, J_z, [I_3]$
- 2body problem $E, \vec{J}$
- Nbody

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Something more:

$$\vec{e}\cdot\vec{r}+r = \frac{\vec{r}(\vec{v}\times\vec{j})}{GM} = \frac{(\vec{r}\times\vec{v})\vec{j}}{GM} = \frac{j^2}{GM}$$

$$r+\vec{e}\vec{r} = er\cos f + r = r(1+e\cos f)$$

$$r = \frac{j^2/GM}{(1+e\cos f)}$$ conic section

$$r_{min} = \frac{j^2/GM}{(1+e)}, \quad r_{max} = \frac{j^2/GM}{(1-e)}$$

$$\cancel{2a = r_{min}(1+e) + r_{max}(1+e)}$$

$$a = (r_{min}+r_{max})/2 = \frac{j^2}{2GM}\left(\frac{1}{1+e}+\frac{1}{1-e}\right)$$

$$= \frac{j^2}{GM(1-e^2)} \Rightarrow \underline{j^2 = GMa(1-e^2)}$$

$$r_{min} = a(1-e) \qquad r_{max} = a(1+e)$$

$\rightarrow E$

4a

Energy, e.g. at pericenter:

$$\frac{E}{\mu} = \frac{1}{2} v^2 - \frac{GM}{r} = \frac{1}{2}\frac{J^2}{r_{min}^2} - \frac{GM}{r_{min}}$$

$$= \frac{GMa(1-e^2)}{2 a^2 (1-e)^2} - \frac{GM}{a(1-e)}$$

$$= \frac{GM}{2a}\left(\frac{1+e}{1-e} - \frac{2}{(1-e)}\right)$$

$$\frac{-1+e}{1-e} = -\frac{1-e}{1-e}$$

$$\frac{E}{\mu} = -\frac{GM}{2a}$$

$$E = -\frac{Gm_1 m_2}{2a}$$

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At r_{min}: (Show that $\vec{e}$ points to pericenter)

$$\vec{e}\vec{r} = \frac{J^2}{GM} - r_{min}$$

$$= \frac{J^2}{GM} - \frac{J^2}{GM(1+e)}$$

$$= \frac{J^2}{GM}\left(1 - \frac{1}{1+e}\right)$$

$$\frac{e}{1+e}$$

$$\left(\vec{e}\cdot\frac{\vec{r}_{min}}{r_{min}}\right)\cdot r_{min} \quad \frac{\cancel{J}^2}{\cancel{GM}}\ \frac{}{1+e}$$

$$\boxed{\vec{e}\cdot\frac{\vec{r}_{min}}{r_{min}} = e}$$

$$|e|\left|\frac{\vec{r}_{min}}{r}\right|\cos(\vec{e},\vec{r}_{min}) = e$$ — angle 0!

5b

Relation to classical orbital elements:

We have $E, \vec{j}$, Pos. of Pericenter. (5)

(Note $\vec{e}, \vec{j}$ not independent: $\vec{e} \cdot \vec{j} = 0$)

$$i = \arccos \frac{j_z}{j} \iff \cos i = \frac{j_z}{j}$$

Inclination

$$\vec{n} = \vec{e}_z \times \vec{j} = (-j_y, j_x, 0) = (n_x, n_y, 0)$$

$$\Omega = \arccos \frac{n_x}{|\vec{n}|} \iff \cos \Omega = \frac{n_x}{|\vec{n}|}$$

$n_y \geq 0$

$$2\pi - \Omega = \arccos \frac{n_x}{|\vec{n}|} \iff \cos(2\pi - \Omega) = \frac{n_x}{|\vec{n}|}$$

$n_y < 0$

Longitude of Ascending Node

$$\omega = \arccos \frac{\vec{n} \cdot \vec{e}}{|\vec{n}| e} \iff \cos \omega = \frac{\vec{n} \cdot \vec{e}}{|\vec{n}| e}$$

Argument of Periapsis